Further Generalization of Taylor Formula and Application

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Abstract: Taylor formula is extensively useful in several fields, such as mathematics, science and technology etc. Because of the wide range of application of Taylor formula, its significance is generally identified. However, Taylor formula is still restricted in vector. In this paper, the domain of Taylor formula is generalized and then range is also generalized. Finally, some specific examples in physics are given.

1. Expansion of the Taylor formula

In general, the Taylor formula for a single-variable function $f(x)$ that is defined in the neighborhood of point x_0 and n times differentiable at x_0 is expressed as

$$f(x)=\sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k+R_{n+1} \tag{1}$$

$f^{(k)}(x_0)$ is the function value of the k-th derivative of $f(x)$ at x_0. R_{n+1} is the remainder of the Taylor formula (The domain and value range of the function discussed in this article only consider real numbers.).

In the multivariate function

$$f(\vec{\boldsymbol{x}})=f(x_1,x_2\dots x_m)=\sum_{k=0}^{n}\frac{1}{k!}\left(\Delta x_1\frac{\partial}{\partial x_1}+\dots+\Delta x_m\frac{\partial}{\partial x_m}\right)^k f(x_{01},x_{02}\dots x_{0m})+R_{n+1} \tag{2}$$

$\Delta x_i=x_i-x_{0i}$. $f(x_{01},x_{02}\dots x_{0m})$ is the value of n times partial differentiable function $f(x_1,x_2\dots x_m)$ at $x_{01},x_{02}\dots x_{0m}$.

As we all know, the general form of a multivariate polynomial is $\sum_{j_1 j_2\cdots j_m} a_{j_1 j_2\cdots j_m} x_1^{j_1}x_2^{j_2}\dots x_m^{j_m}$, x_1, $x_2\dots x_m$ are m indeterminates, $j_1,j_2\dots j_m$ is the degree of each indeterminate, $j_1+j_2+\dots j_m$ is the degree of the monomial, and $a_{j_1 j_2\cdots j_m}$ is the coefficient of monomial when the degree of x_1 is j_i, the degree of x_2 is j_2 …and the degree of x_m is j_m. Its tensor representation is

$\mathrm{P}[\vec{\boldsymbol{x}}]=A_0+\overleftrightarrow{\boldsymbol{A}}_1\cdot\vec{\boldsymbol{x}}+\overleftrightarrow{\boldsymbol{A}}_2:(\vec{\boldsymbol{x}}\otimes\vec{\boldsymbol{x}})+\overleftrightarrow{\boldsymbol{A}}_3\vdots(\vec{\boldsymbol{x}}\otimes\vec{\boldsymbol{x}}\otimes\vec{\boldsymbol{x}})+\dots+\overleftrightarrow{\boldsymbol{A}}_{\boldsymbol{n}}^{(n)}\cdot^{(n)}\vec{\boldsymbol{x}}^{\otimes n}$. A_0 is a 0th order tensor, i.e. a constant, $\overleftrightarrow{\boldsymbol{A}}_1$ is an m-dimensional vector, $\overleftrightarrow{\boldsymbol{A}}_2$ is an 2nd order symmetric matrix, and $\overleftrightarrow{\boldsymbol{A}}_{\boldsymbol{n}}$ is an n-th order square tensor (i.e. the positive integer set of each index is the same); $\vec{\boldsymbol{x}}=(x_1,x_2,\dots,x_m)^T$ is the vector of the respective variables, $\otimes$ denotes the tensor product, $\vec{\boldsymbol{x}}^{\otimes n}=\vec{\boldsymbol{x}}\otimes\vec{\boldsymbol{x}}\otimes$(n times multiplication…)$\otimes\vec{\boldsymbol{x}}$, and $\cdot^{(n)}$ means n times dot product.[3]

Combining the ideas of the tensor representation of the above polynomial, we point out:

Theorem 1: The tensor representation of the multivariate Taylor formula is:

$$f\left(\vec{x}\right)=\sum_{k=0}^{n}\frac{1}{k!}\nabla^{\otimes k} f\left(\overrightarrow{x_0}\right)\cdot^{(k)}\left(\vec{x}-\overrightarrow{x_0}\right)^{\otimes k}+R_{n+1} \tag{3}$$

and is equivalent to (2).

Proof: When n＝0, it is clearly established. When n＝1,

$$f\left(\vec{x}\right) = f\left(\overrightarrow{x_0}\right)+\nabla f\left(\vec{x}\right)|_{\vec{x}=\overrightarrow{x_0}}\cdot\left(\vec{x}-\overrightarrow{x_0}\right)+R_2$$

$$= f\left(x_{01}, x_{02}\dots x_{0m}\right)+\left(f'_{(x_{01})}, f'_{(x_{02})},\dots, f'_{(x_{0m})}\right)\cdot\left(x_1-x_{01}, x_2-x_{02},\dots, x_m-x_{0m}\right)+R_2$$

$$= f\left(x_{01}, x_{02}\dots x_{0m}\right)+f'_{(x_{01})}\left(x_1-x_{01}\right)+\dots+f'_{(x_{0m})}\left(x_m-x_{0m}\right)+R_2$$

In the traditional Taylor formula representation

$$f\left(x_1,x_2\dots x_m\right)= f\left(x_{01},x_{02}\dots x_{0m}\right)+\left(\Delta x_1\frac{\partial}{\partial x_1}+\dots+\Delta x_m\frac{\partial}{\partial x_m}\right)f\left(x_{01},x_{02}\dots x_{0m}\right)+R_2$$

$= f'_{(x_{01})}\left(x_1-x_{01}\right)+\dots+f'_{(x_{0m})}\left(x_m-x_{0m}\right)+R_2$, equal to each other.

Then for the general case n, you only need to consider the n-th item. The n-th item of the tensor notation is $\frac{1}{n!}\nabla^{\otimes n} f\left(\overrightarrow{x_0}\right)\cdot^{(n)}\left(\vec{x}-\overrightarrow{x_0}\right)^{\otimes n}$, we write it in component form (index form):

$$\frac{1}{n!}\frac{\partial}{\partial x_{i_1}}\frac{\partial}{\partial x_{i_2}}\dots\frac{\partial}{\partial x_{i_n}} f\,\Delta x_{i_1}\Delta x_{i_2}\dots\Delta x_{i_n}=\frac{1}{n!}\left(\Delta x_1\frac{\partial}{\partial x_1}+\dots+\Delta x_m\frac{\partial}{\partial x_m}\right)\frac{\partial}{\partial x_{i_2}}\dots\frac{\partial}{\partial x_{i_n}} f\,\Delta x_{i_2}\dots\Delta x_{i_n}$$

$$=\frac{1}{n!}\left(\Delta x_1\frac{\partial}{\partial x_1}+\dots+\Delta x_m\frac{\partial}{\partial x_m}\right)^2\frac{\partial}{\partial x_{i_3}}\dots\frac{\partial}{\partial x_{i_n}} f\,\Delta x_{i_3}\dots\Delta x_{i_n}=\frac{1}{n!}\left(\Delta x_1\frac{\partial}{\partial x_1}+\dots+\Delta x_m\frac{\partial}{\partial x_m}\right)^n f\left(x_{01},x_{02}\dots x_{0m}\right)$$

, which is equal to the n-th term in traditional notation. The remainder R_{n+1} has nothing to do with the discussion above, so it is proved.

Note: The indices representation of (3) is

$$f\left(x_i\right)=\sum_{k=0}^{n}\frac{1}{k!}\frac{\partial}{\partial x_{i_1}}\frac{\partial}{\partial x_{i_2}}\dots\frac{\partial}{\partial x_{i_k}} f\,|_{x_1=x_{01},\dots,x_m=x_{0m}}\;\Delta x_{i_1}\Delta x_{i_2}\dots\Delta x_{i_k}+R_{n+1} \tag{3'}$$

In fact, from another perspective, the essence of the Taylor formula is to use the polynomial to simulate the properties of a function around a particular point, and because of the equivalence of the multivariate polynomial and the tensor, it is obvious that the tensor representation is reasonable.

2. Domain's generalization and high-order derivatives

For the definition of higher-order derivatives and differentiations of analogous single-variable functions, we can define the k-th derivative of the multivariate function $f\left(\vec{x}\right)$ at the constant vector $\overrightarrow{x_0}$ as $f^{(k)}_{(\overrightarrow{x_0})}=\nabla^{\otimes k} f\left(\overrightarrow{x_0}\right)$, of course, its k-th derivative is $f^{(k)}_{(\vec{x})}=\nabla^{\otimes k} f\left(\vec{x}\right)$. Its first derivative is $f'_{(\vec{x})}=\nabla f\left(\vec{x}\right)$,which is the gradient we are familiar with. It is easy to get the differentiation of $f\left(\vec{x}\right)$, i.e. $\mathrm{d}f\left(\vec{x}\right)=\nabla f\left(\vec{x}\right)\cdot\mathrm{d}\vec{x}$, $\mathrm{d}\vec{x}=\left(\mathrm{d}x_1,\mathrm{d}x_2,\dots,\mathrm{d}x_m\right)^T$.

The above result can be generalized to a vector value function by a very simple step, that is, the function value is a vector, $f:\mathbb{R}^m\to\mathbb{R}^p$, $\vec{y}=\vec{f}\left(\vec{x}\right)$. Obviously any component y_j of $\vec{y}$ has a relationship with $y_j=f_j\left(\vec{x}\right)$, then using the Taylor formula, we can get $y_j=\sum_{k=0}^{n}\frac{1}{k!}\nabla^{\otimes k} f_j\left(\overrightarrow{x_0}\right)\cdot^{(k)}\left(\vec{x}-\overrightarrow{x_0}\right)^{\otimes k}+R_{j,n+1}$. This can also be written in the form of a column vector:

$$\vec{y}=\begin{pmatrix} y_i \\ y_2 \\ \vdots \\ y_p \end{pmatrix}=\sum_{k=0}^{n}\begin{pmatrix} \frac{1}{k!}\nabla^{\otimes k} f_1\left(\overrightarrow{x_0}\right)\cdot^{(k)}\left(\vec{x}-\overrightarrow{x_0}\right)^{\otimes k} \\ \frac{1}{k!}\nabla^{\otimes k} f_2\left(\overrightarrow{x_0}\right)\cdot^{(k)}\left(\vec{x}-\overrightarrow{x_0}\right)^{\otimes k} \\ \vdots \\ \frac{1}{k!}\nabla^{\otimes k} f_p\left(\overrightarrow{x_0}\right)\cdot^{(k)}\left(\vec{x}-\overrightarrow{x_0}\right)^{\otimes k} \end{pmatrix}+\overrightarrow{R_{n+1}}=\sum_{k=0}^{n}\begin{pmatrix} \frac{1}{k!}\nabla^{\otimes k} f_1\left(\overrightarrow{x_0}\right) \\ \frac{1}{k!}\nabla^{\otimes k} f_2\left(\overrightarrow{x_0}\right) \\ \vdots \\ \frac{1}{k!}\nabla^{\otimes k} f_p\left(\overrightarrow{x_0}\right) \end{pmatrix}\cdot^{(k)}\left(\vec{x}-\overrightarrow{x_0}\right)^{\otimes k}+\overrightarrow{R_{n+1}}$$

$$=\sum_{k=0}^{n}\frac{1}{k!}\nabla^{\otimes k}\vec{f}\left(\vec{x}\right)|_{\vec{x}=\overline{x_0}}\cdot^{(k)}\left(\vec{x}-\overrightarrow{x_0}\right)^{\otimes k}+\overrightarrow{R_{n+1}} \quad (4)$$

Its indices representation is：

$$y_j=\sum_{k=0}^{n}\frac{1}{k!}\frac{\partial}{\partial x_{i_1}}\frac{\partial}{\partial x_{i_2}}\cdots\frac{\partial}{\partial x_{i_k}} f_j\left(x_{01},x_{02}\ldots x_{0m}\right)\Delta x_{i_1}\Delta x_{i_2}\ldots\Delta x_{i_k}+R_{j,n+1} \quad (4').$$

In order to get a more compact structure, let's observe its regulation. When n＝0, $\vec{y}=\vec{f}\left(\overrightarrow{x_0}\right)+\overrightarrow{R_1}$.

When n＝1,

$$\vec{y}=\vec{f}\left(\overrightarrow{x_0}\right)+\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_p} \\ \vdots & \vdots & \vdots & 0 \\ \frac{\partial f_p}{\partial x_1} & \cdots & \cdots & \frac{\partial f_p}{\partial x_m} \end{pmatrix}\begin{pmatrix} x_1-x_{01} \\ x_2-x_{02} \\ \vdots \\ x_m-x_{0m} \end{pmatrix}+\overrightarrow{R_2}$$

.Combine the above ideas,

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_p} \\ \vdots & \vdots & \vdots & 0 \\ \frac{\partial f_p}{\partial x_1} & \cdots & \cdots & \frac{\partial f_p}{\partial x_m} \end{pmatrix}$$

can be defined as the first derivative of a vector-valued function, which is the Jacobian matrix. We can write it down as $\overleftrightarrow{D_f}=\nabla\otimes\vec{f}\left(\vec{x}\right)$. Obviously, the k-th derivative of $\vec{f}\left(\vec{x}\right)$ is $\vec{f}^{(k)}_{(\vec{x})}=\nabla^{\otimes k}\otimes\vec{f}\left(\vec{x}\right)$, and its differentiation is $\mathrm{d}\vec{f}\left(\vec{x}\right)=\overleftrightarrow{D_f}\mathrm{d}\vec{x}$.

3. Further generalization

After the above discussion, we can try to further generalize the domain of the function to the tensor set, i.e. $f:R^{m^n}\to R$, $y=f\left(\vec{x}^{(n)}\right)$. Here has been simplified, Set the argument $\vec{x}^{(n)}$ as a square tensor (the dimension of each index is equal, that is to say, the n indices of the component can only be taken from 1 to m). If it is a general tensor, it is more troublesome to discuss. Interested readers can try it for themselves.

Tensor $\vec{x}^{(n)}=x_{i_1i_2\ldots i_n}\hat{e}_{i_1}\hat{e}_{i_2}\ldots\hat{e}_{i_n}$, representing linear combination of tensor basis $\hat{e}_{i_1}\hat{e}_{i_2}\ldots\hat{e}_{i_n}$ with coefficient $x_{i_1 i_2 \ldots i_n}$. Define differential operator $\vec{\boldsymbol{D}}^{(n)} = \frac{\partial}{\partial x_{i_1} x_{i_2} \ldots x_{i_n}} \hat{\boldsymbol{e}}_{i_1} \hat{\boldsymbol{e}}_{i_2} \ldots \hat{\boldsymbol{e}}_{i_n}$. For convenience, $\frac{\partial}{\partial x_{i_1 i_2 \ldots i_n}}$ is simply denoted as $\partial_{i_1 i_2 \ldots i_n}$.

The differential in general case is $\mathrm{d}f\left(\vec{\boldsymbol{x}}^{(n)}\right) = \vec{\boldsymbol{D}}^{(n)} f\left(\vec{\boldsymbol{x}}^{(n)}\right) \cdot^{(n)} \mathrm{d}\vec{\boldsymbol{x}}^{(n)} = (\partial_{i_1 i_2 \ldots i_n} f) dx_{i_1 i_2 \ldots i_n}$, and the k-th derivative is naturally $f^{(k)}_{\left(\vec{x}^{(n)}\right)} = \vec{\boldsymbol{D}}^{(n)\otimes k} f\left(\vec{\boldsymbol{x}}^{(n)}\right)$.

Therefore, a function $f\left(\vec{\boldsymbol{x}}^{(n)}\right)$ that is s times differentiable for any $x_{i_1 i_2 \ldots i_n}$ has a corresponding Taylor's formula

$$f\left(\vec{\boldsymbol{x}}^{(n)}\right) = \sum_{k=0}^{s} \frac{1}{k!} \vec{\boldsymbol{D}}^{(n)\otimes(k)} \otimes f\left(\overleftrightarrow{\boldsymbol{x}_0}^{(n)}\right) \cdot^{(nk)} \left(\vec{\boldsymbol{x}}^{(n)} - \overleftrightarrow{\boldsymbol{x}_0}^{(n)}\right)^{\otimes k} + \overrightarrow{\boldsymbol{R}_{s+1}} \quad (5)$$

If you want to write a component representation and find that the indices are not enough. We can replace it with a combination of Roman numerals, uppercase letters and positive integers:

$$f\left(x_{i_1 i_2 \ldots i_n}\right) = \sum_{k=0}^{s} \frac{1}{k!} \partial_{\mathrm{I_1 I_2 \ldots I_n}} \partial_{\mathrm{II_1 II_2 \ldots II_n}} \ldots \partial_{\mathrm{K_1 K_2 \ldots K_n}} f\left(x_{0, i_1 i_2 \ldots i_n}\right) \Delta x_{\mathrm{I_1 I_2 \ldots I_n}} \Delta x_{\mathrm{II_1 II_2 \ldots II_n}} \ldots \Delta x_{\mathrm{K_1 K_2 \ldots K_n}} + R_{s+1, i_1 i_2 \ldots i_n} \quad (5')$$

This is the Taylor formula of an arbitrary function of tensor.

Using the previously method of generalizing range, it is possible to generalize the function value to a tensor to obtain the most general case. $f: \mathbb{R}^{m^n} \to \mathbb{R}^{p^q}$, $\vec{\boldsymbol{y}}^{(q)} = \vec{f}^{(q)}\left(\vec{\boldsymbol{x}}^{(n)}\right)$, the meaning is that the element in the definition domain is an n-th order tensor with each index from 1 to m. The elements in the range are q-th order tensors with each index from 1 to p.

Through analogical reasoning, we can see that the first derivative is $\vec{\boldsymbol{D}}^{(n)} \otimes \vec{f}^{(q)}\left(\vec{\boldsymbol{x}}^{(n)}\right)$, the differentiation is $\mathrm{d}\vec{f}^{(q)} = \left(\vec{\boldsymbol{D}}^{(n)} \otimes \vec{f}^{(q)}\right) \cdot^{(n)} \boldsymbol{d}\vec{\boldsymbol{x}}^{(n)}$, and the k-th derivative is naturally $\vec{f}^{(q)[k]}\left(\vec{\boldsymbol{x}}^{(n)}\right) = \vec{\boldsymbol{D}}^{(n)\otimes(k)} \otimes \vec{f}^{(q)}$.

So immediately write the Taylor formula of the tensor value function：

$$\vec{\boldsymbol{f}}^{(q)} = \sum_{k=0}^{s} \frac{1}{k!} \vec{\boldsymbol{D}}^{(n)\otimes(k)} \otimes \vec{f}^{(q)}\left(\overleftrightarrow{\boldsymbol{x}_0}^{(n)}\right) \cdot^{(nk)} \left(\vec{\boldsymbol{x}}^{(n)} - \overleftrightarrow{\boldsymbol{x}_0}^{(n)}\right)^{\otimes k} + \overleftrightarrow{\boldsymbol{R}_{s+1}} \quad (6)$$

The indices representation:

$$f^{(q)}_{j_1 j_2 \ldots j_q}\left(x_{i_1 i_2 \ldots i_n}\right) = \sum_{k=0}^{s} \frac{1}{k!} \partial_{\mathrm{I_1 I_2 \ldots I_n}} \partial_{\mathrm{II_1 II_2 \ldots II_n}} \ldots \partial_{\mathrm{K_1 K_2 \ldots K_n}} f^{(q)}_{j_1 j_2 \ldots j_q}\left(x_{i_1 i_2 \ldots i_n}\right) \Delta x_{\mathrm{I_1 I_2 \ldots I_n}} \Delta x_{\mathrm{II_1 II_2 \ldots II_n}} \ldots \Delta x_{\mathrm{K_1 K_2 \ldots K_n}} \quad (6')$$

4. Applications

Example 1: In a classical charged particles system, the electric potential φ of the system can be expressed as integral

$$\varphi\left(\vec{\boldsymbol{x}}\right) = k \int_{V'} \frac{\rho\left(\vec{\boldsymbol{x}}'\right) dV'}{\left|\vec{\boldsymbol{x}} - \vec{\boldsymbol{x}}'\right|} \quad (7)$$

In $\mathbb{R}^3$, $\vec{\boldsymbol{x}}$ is the field point coordinates, $\vec{\boldsymbol{x}}'$ is the source point coordinates, where k is the electrostatic constant, and $\rho\left(\vec{\boldsymbol{x}}'\right)$ is a function of charge density with respect to the position of the source.

We use Taylor's formula to expand $f\left(\vec{x}'\right)=\frac{1}{\left|\vec{x}-\vec{x}'\right|}$ at $\vec{x}'=\vec{0}$ to a power series of $\vec{x}'$.

$f\left(\vec{0}\right)=\frac{1}{\left|\vec{x}\right|}$, $f'\left(\vec{0}\right)=\nabla\frac{1}{\left|\vec{x}-\vec{x}'\right|}|_{\vec{x}'=\vec{0}}=-\frac{1}{\vec{x}^2}=-\nabla\frac{1}{\left|\vec{x}\right|}$, $f''_{\left(\vec{0}\right)}=\nabla\otimes\nabla\frac{1}{\left|\vec{x}-\vec{x}'\right|}|_{\vec{x}'=\vec{0}}=\nabla\otimes\nabla\frac{1}{\left|\vec{x}\right|}$.

Obviously,

$f^{(n)}_{\left(\vec{0}\right)}=\nabla^{\otimes n}\frac{1}{\left|\vec{x}-\vec{x}'\right|}|_{\vec{x}'=\vec{0}}=\left(-1\right)^n\nabla^{\otimes n}\frac{1}{\left|\vec{x}\right|}$. Use (3) and then brings the result into (7), so:

$$\varphi\left(\vec{x}\right)=k\int_{V'}\rho\left(\vec{x}'\right)\frac{1}{\left|\vec{x}\right|}dV'-k\int_{V'}\rho\left(\vec{x}'\right)\nabla\frac{1}{\left|\vec{x}\right|}\cdot\vec{x}'dV'+\ldots+\left(-1\right)^n k\int_{V'}\rho\left(\vec{x}'\right)\nabla^{\otimes n}\frac{1}{\left|\vec{x}\right|}\cdot^{(n)}\vec{x}'^{\otimes n}dV'+\ldots \quad (8)$$

We call the n-th term of the electric potential of the electric 2^{n-1} moment. For example, the second term is the electric dipole moment potential and the third term is the electric quadrupole moment potential. With the same reason, the magnetic vector potential $\vec{A}$ can also be expanded into the sum of the magnetic multipole moments.

Example 2: In fluid mechanics, the velocity $\vec{v}$ about a specific point $\vec{r}$ in $\mathbb{R}^3$, $\vec{v}\left(\vec{r}\right)$ is called the velocity field in space, which is generally considered to be continuous. We can use the Taylor formula to analyze the properties of velocity in a neighborhood of any given point in the field. Using (4), get $\vec{v}\left(\vec{r}+\Delta\vec{r}\right)=\vec{v}\left(\vec{r}\right)+\nabla\otimes\vec{v}\cdot\Delta\vec{r}+\vec{R}_2$. Look at each item in order, where the first item $\vec{v}\left(\vec{r}\right)$ represents the translation of the fluid. As for the second item, obviously it is a 3×3 matrix, we may wish to write it as a symmetric matrix and an antisymmetric matrix sum $\nabla\otimes\vec{v}=\frac{1}{2}\left[\nabla\otimes\vec{v}+\left(\nabla\otimes\vec{v}\right)^T\right]+\frac{1}{2}\left[\nabla\otimes\vec{v}-\left(\nabla\otimes\vec{v}\right)^T\right]$, define $\vec{\varepsilon}^{(2)}=\frac{1}{2}\left[\nabla\otimes\vec{v}+\left(\nabla\otimes\vec{v}\right)^T\right]$ as the strain rate matrix (tensor). And the antisymmetric term, there existence relationship $\frac{1}{2}\Delta\vec{r}\cdot\left[\nabla\otimes\vec{v}-\left(\nabla\otimes\vec{v}\right)^T\right]=\frac{1}{2}\nabla\times\vec{v}\times\Delta\vec{r}$, is the rotation rate. Therefore, we have

$$\vec{v}\left(\vec{r}+\Delta\vec{r}\right)=\vec{v}\left(\vec{r}\right)+\vec{\varepsilon}^{(2)}\cdot\Delta\vec{r}+\frac{1}{2}\nabla\times\vec{v}\times\Delta\vec{r}+\vec{R}_2 \quad (9)$$

The first three terms represent the translation, shearing motion and rotation of the fluid.

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